Homework 0

Due: 9 January, 2026

This homework is on prerequisite knowledge that you are expected to have/pick up before the course starts. Even though this homework is not graded, you should submit it so that we can make sure that you have all the prerequisite knowledge to succeed in this course.

Problem 1

Show that for any $\mathbb{R}$-valued random variable X, any natural number k and any $a > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)|^k > a^k) \leq \frac{\mathbb{E}(|X - \mathbb{E}(X)|^k)}{a^k}$$

This is a generalisation of Chebyshev's inequality. The quantity $\mathbb{E}(|X - \mathbb{E}(X)|^k)$ is called the k^{th} centred moment of the variable X, at least for even k. For odd numbers sometimes people use $\mathbb{E}((X - \mathbb{E}(X))^k)$ and sometimes $\mathbb{E}(|X - \mathbb{E}(X)|^k)$, but the inequality only holds for the absolute value version. Why is that?

Problem 2

Recall that the moment generating function of a random variable X is defined as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right)$$

Suppose X_i are independently and identically distributed (i.i.d.) random variables, and let $\bar{X}_n = (\sum_1^n X_i)/n$.

Show that

$$M_{\bar{X}_n}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n$$

Problem 3

Suppose event A happens with probability at least $1 - \delta$, and that the probability of event B happening conditioned on event A is at least $1 - \epsilon$, where δ and ϵ are very small (perhaps in some asymptotic regime).

Show that the probability of A and B happening is at least $1 - (\delta + \epsilon)$.

Problem 4

Given two matrices A and B (of the appropriate sizes for multiplication), what is $(AB)_{ij}$ as a dot product of row and column vectors of A and B?

Problem 5

Show that for any $x \in \mathbb{R}$, $1 + x \leq e^x$, and equivalently (for $x > -1$), $\log(1 + x) \leq x$.

Show that $\frac{1}{1+x} = 1 - O(x)$ where the big-O is for the limit $x \to 0$. Similarly, show that $\frac{1}{1-x} = 1 + O(x)$ where the big-O is for the limit $x \to 0^+$ (i.e. x approaches 0 from the positive side).

(For the last statement, you can show something similar for $x \to 0^-$, but with a *different* hidden constant in the big-O.)

(Hint: For the second part, first write out a precise statement to prove, of the form "if $x \leq$?? (or if $|x| \leq$??), then some inequality is true" for some appropriate constant "??". The rest of the problem is just calculus or Taylor expansions, that admits short proofs.)

Problem 6

Suppose there are n numbers $x_1, \ldots, x_n$ are such that $\sum x_i \leq 1$. Show that there must exists an i such that $x_i \leq \frac{1}{n}$.

Furthermore, suppose there is another sequence of n numbers $y_1, \ldots, y_n$ such that $\sum y_i = 1$ (note the equality here, not an inequality). Show that there must exists an i such that $x_i \leq y_i$.